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On wave propagation in viscoelastic media with concave creep compliance

Andrzej Hanyga
ul. Bitwy Warszawskiej 1920 r. 14/52
02-366 Warszawa, Poland

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Abstract

It is proved that the attenuation function of a viscoelastic medium with a non-decreasing and concave creep compliance is sublinear in the high frequency range.

Keywords.
viscoelasticity, wave attenuation, creep compliance, completely monotonic, Bernstein function, creep function, bio-tissue, polymer

Notation.

$\mathbb{R}$	the set of real numbers	
$\mathbb{C}$	the set of complex numbers	
$f \circ g$	composition	$(f \circ g)(x) := f(g(x))$
$\mathcal{L}[f](p) = \tilde{f}(p)$	Laplace transform of $f(t)$	$\tilde{f}(p) = \int_0^\infty \mathrm{e}^{-pt} f(t)\,\mathrm{d}t$
$\hat{f}(k)$	Fourier transform of $f(x)$	$\hat{f}(k) = \int_{-\infty}^\infty \mathrm{e}^{-\mathrm{i}k\cdot x} f(x)\,\mathrm{d}x$

1 Introduction.

Ultrasonic tests of mechanical properties of polymers and bio-tissues assume that the material is viscoelastic. In applications a linear viscoelastic material is always assumed to have a completely monotonic (CM) relaxation modulus [1, 2, 3]. This assumption has important consequences for the attenuation and dispersion of acoustic waves. In particular, the increase of attenuation with frequency in the high frequency range is slower than linear [4]. This fact cannot be verified experimentally because experimental test cover the range of frequencies below 250 MHz. The results of ultrasonic tests of polymers and bio-tissues suggest a power-law dependence of attenuation in the frequency range accessible to measurements, $\mathcal{A}(\omega) = A\,\omega^\alpha$ with $\alpha > 1$. This power-law dependence is then extrapolated to high frequency range, where it contradicts various general

principles. It is then explained in terms of *ad hoc* lossy wave equations [5, 6, 7] and wave equations with fractional Laplacians [15, 16] which are incompatible with viscoelastic constitutive equations.

Superlinear growth of attenuation in the high frequency range is inconsistent with finite speed of propagation. The requirement of finite speed of propagation imposes an upper growth limit $\omega/\ln(\omega)$ on high-frequency attenuation: the propagation speed is finite if $\omega/\ln^{\alpha}(\omega)$, $\alpha > 1$ but not for $\alpha \leq 1$. Superlinear growth of attenuation in the high frequency range is also inconsistent with viscoelastic constitutive equations even if the propagation speed is infinite, for example it is inconsistent with Newtonian viscosity, where the attenuation satisfies the power law const $\times\, \omega^{1/2}$, and with strongly singular locally integrable completely monotonic relaxation moduli [4].

According to the theory of wave propagation in viscoelastic media with completely monotonic relaxation moduli, developed in [8, 4], in the low frequency range the wave attenuation in a viscoelastic solid increases like $\omega^{1+\alpha}$, $0 < \alpha \leq 1$, while a viscoelastic fluid is characterized by a rate of increase ω^{α}, $0 < \alpha < 1$. Low frequency behavior of attenuation is accessible to measurements and is consistent with observations. In the high frequency range the growth of attenuation is however sublinear. High frequency attenuation is also sublinear in viscoelastic media with spatial derivative operators of fractional order [14].

Although practical modeling of viscoelastic media routinely assumes that the relaxation modulus is completely monotonic, there is hardly any a priori evidence for this assumption. The only general physical principle – the fluctuation-dissipation theorem – implies that the relaxation modulus is completely positive [9]. The consequences of complete positivity for the dependence of attenuation on frequency are however at present unknown. In this paper we explore the consequences of a different extension of the complete monotonicity assumption, based on the assumption that the creep compliance is non-decreasing and concave. Non-negative functions with these two properties are said to belong to the CrF class, which is an abbreviation of Prüss' unfortunate term "creep functions". The above assumption, proposed and explored in Jan Prüss' book [10], is much weaker than the hypothesis of a completely monotonic relaxation modulus, assumed in [8, 4]. The latter is equivalent to the creep compliance being a Bernstein function, which is defined in terms of an infinite sequence of inequalities for the derivatives of the creep compliance of arbitrary order. In the CrF class only the three first of these inequalities are required to hold. The Bernstein class and the CrF class includes Newtonian viscosity characterized by linear creep. Unlike the Bernstein property, the CrF property of a function can be easily visually recognized from its plot. Furthermore, the CrF property is stable with respect to perturbations which are small in the sup norm.

It will be shown here that wave attenuation in media with CrF creep compliances is bounded by a function $a + b\,|\omega|$. A sharper form of this bound was obtained for creep compliances in the Bernstein class in [8, 4]. The theory developed in [8, 4] relies on a spectral analysis of attenuation and dispersion. Unfortunately a spectral analysis of attenuation and dispersion similar to the completely monotonic case has not been possible but it is very likely that the

bound obtained here for the CrF class of creep compliances can be significantly sharpened.

After a recapitulation of necessary facts from the theory of CM functions and Bernstein functions (Sec. 2) the essentials of Prüss' theory are recalled in Sec. 3. The main result concerning the bound on wave attenuation is derived in Sec. 5.

2 Some properties of completely monotonic and Bernstein functions.

Definition 2.1. *A real function f on $]0,\infty[$ is said to be completely monotonic (CM) if $f \in \mathcal{C}^n$ and $(-1)^n \,\mathrm{D}^n\, f(x) \geq 0$ for for every integer $n \geq 0$.*

Theorem 2.2. *(Bernstein)*
A real function f on $]0,\infty[$ is CM if and only if there is a positive Radon measure μ on $[0,\infty[$ such that

$$f(t) = \int_{[0,\infty[} \mathrm{e}^{-tr} \mu(\mathrm{d}r), \qquad t > 0 \tag{1}$$

A CM function is locally integrable if and only it is integrable over the interval $[0,1]$. The set of locally integrable CM (in short LICM) functions is denoted by $\mathfrak{L}$.

Theorem 2.3. *[11, 4]*
A real function f is LICM if and only if there is a positive Radon measure μ such that

$$\int_{[0,\infty[} \frac{\mu(\mathrm{d}r)}{1+r} < \infty \tag{2}$$

and eq. (1) *holds.*

Definition 2.4. *A real function f defined on $[0,\infty[$ is said to be a Bernstein function (BF) if it has a derivative which is a LICM function.*
The set of Bernstein functions is denoted by the symbol $\mathfrak{B}$.

It is obvious that a Bernstein function is non-negative, non-decreasing and concave.

From Bernstein's theorem the following representation of Bernstein functions can be derived:

Theorem 2.5. *[12]*
A function $f : \mathbb{R}_+ \rightarrow \mathbb{R}$ is a Bernstein function if and only if the are two non-negative real numbers a, b and a positive Radon measure λ on $\mathbb{R}_+$ such that

$$\int_{]0,\infty[} \frac{r}{1+r}\, \lambda(\mathrm{d}r) < \infty \tag{3}$$

and

$$f(t) = a + b\,t + \int_{]0,\infty[} \left[1 - \mathrm{e}^{-rt}\right]\, \lambda(\mathrm{d}r), \qquad t > 0 \tag{4}$$

Inequality (3) is equivalent to the following pair of inequalities

$$\int_{]0,1]} r\nu(\mathrm{d}r) < \infty, \qquad \int_{]1,\infty[} \nu(\mathrm{d}r) < \infty \tag{5}$$

Theorem 2.6. *If f satisfies eq.* (4) *with λ satisfying eq.* (3)*, then* $\lim_{t\to\infty} f(t)/t = b$.

Proof. Expansion in power series shows that $(1+t)^2/2 \le \mathrm{e}^t \le (1+t)^2$ for all $t \ge 0$. Hence $(1+t)^{-2} \le \mathrm{e}^{-t} \le 2(1+t)^{-2}$ for $t \ge 0$ and, upon integration, $t/(1+t) \le 1 - \mathrm{e}^{-t} \le 2t/(1+t)$.

Now

$$\frac{f(t)}{t} = \frac{a}{t} + b + \int_{]0,\infty[} \frac{1 - \mathrm{e}^{-rt}}{t}\, \lambda(\mathrm{d}r) \tag{6}$$

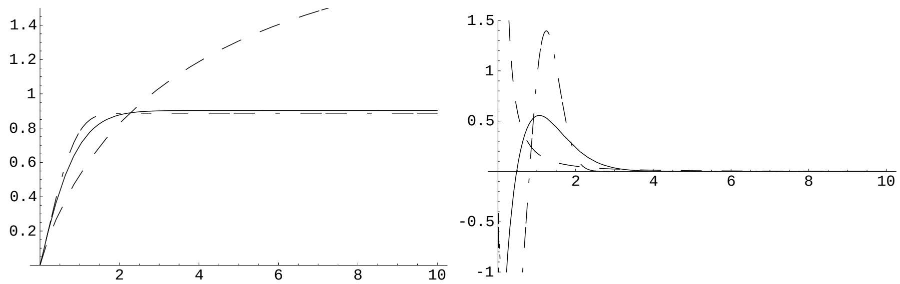
The integrand of the integral on the right-hand side of eq. (6) tends to 0 as $t \to \infty$. The integrand is also bounded from above by $2r/(1+rt)$ and by $2r/(1+r)$ for $t \ge 1$. In view of eq. (3) and the Lebesgue Dominated Convergence Theorem the integral on the right-hand side of eq. (6) tends to 0 as $t \to \infty$. The thesis follows from eq. (6). $\square$

Theorem 2.7. *[13]*
If the real function f defined on $]0,\infty[$ is CM and $g \in \mathfrak{B}$, then the composition $f \circ g$ is CM.
If f, g are Bernstein functions on $[0,\infty[$, then the composition $f \circ g$ is a Bernstein function.
If a positive real function f defined on $[0,\infty[$ is a BF then $1/f$ is CM on $]0,\infty[$.

3 The CrF class of functions.

In every neighborhood of a bounded CM function in the space of bounded continuous functions $\mathcal{C}^0([0,\infty[)$ endowed with the sup norm there is a function g which is not CM, for example $g(x) = f(x) + \sin(x)/n$. Hence the CM property of the relaxation modulus cannot be experimentally confirmed by a direct method although it can be inconsistent with experimental data in a specific case. Thus complete monotonicity has to be considered as an a priori constraint on the interpolation of experimental data by matching them against a multi-parameter family of completely monotonic functions (Prony sums, Cole-Cole or Havriliak-Negami functions, the Kohlrausch-Watts-Williams function).

It is therefore interesting to note that the linear upper bound on the high frequency behavior of the attenuation function is not a consequence of complete monotonicity of relaxation moduli. As an alternative to viscoelasticity based on completely monotonic relaxation functions, a theory based on an experimentally verifiable concavity property of creep compliances will be presented here following [10].



(a) (b)

Figure 1: (a): The function defined by (7), for $\alpha = 1/2$ (dashed), $\alpha = 3/2$ (solid) and $\alpha = 5/2$ (dot-dashed). (b) The 3rd order derivative $\mathrm{D}^3 f$ for the same values of α.

Definition 3.1. *Let I be a segment of the real line.*

A function $f : I \to \mathbb{R}$ is said to be concave *if*

$$\vartheta\, f(x) + (1-\vartheta)\, f(y) \leq f(\vartheta\, x + (1-\vartheta)\, y)$$

for all $x, y \in I$, $0 \leq \vartheta \leq 1$.

Bernstein functions are non-negative, non-decreasing and concave.

Definition 3.2. *A function $f : \mathbb{R}_+ \to \mathbb{R}$ is said to be a* CrF *if it is non-negative, non-decreasing and concave.*

$\mathfrak{Q}$ denotes the set of all the CrFs.

Every Bernstein function is obviously a CrF.

Example 3.3. *The function*

$$f(x) := \int_0^x \mathrm{e}^{-y^\alpha}\, \mathrm{d}y \tag{7}$$

(Fig. 1) is a CrF for $\alpha > 0$, but it is not a BF if $\alpha > 1$. Indeed, if $\alpha > 0$ then $f, f' > 0$ while $f'' < 0$.

It $\alpha \leq 1$ then $f'(x) \equiv \mathrm{e}^{-x^\alpha}$ is CM, hence $f \in \mathfrak{B}$.

If $\alpha > 1$ then $f'''(x) \equiv \alpha\, x^{\alpha-2}\, [\alpha\, x - (\alpha - 1)]\, \mathrm{e}^{-x^\alpha}$ changes sign at $x = (\alpha-1)/\alpha$. Hence f is not a Bernstein function in this case.

Theorem 3.4. *Every CrF f can be expressed in the form*

$$f(t) = a + b\, t + \int_0^t k(s)\, \mathrm{d}s \tag{8}$$

where a, b are non-negative numbers and the locally integrable function k is non-negative, non-increasing and $k(s) \to 0$ for $s \to \infty$.

Proof. Let $a := f(0)$ and $b := \inf_{t>0} f(t)/t$. The function b assumes finite values because concavity

$$f(\vartheta\, t) \geq (1-\vartheta)\, a + \vartheta\, f(t)$$

implies that

$$\frac{f(\vartheta\, t)}{\vartheta\, t} - \frac{1-\vartheta}{\vartheta\, t} a \geq \frac{f(t)}{t}$$

for $t > 0$ and $0 < \vartheta < 1$, hence $f(t)/t$ is decreasing and has a finite limit b. The function f is non-negative and non-decreasing, hence it is bounded on an interval $[0, \varepsilon]$, $\varepsilon > 0$. Since $-f$ is convex, f has a derivative f' at all but a countable set of points. The derivative f' is almost everywhere non-negative (because f is non-decreasing) and almost everywhere non-increasing (because f is concave). The second-order derivative f'' exists and is non-positive almost everywhere. Hence $f'(t) \leq f'(t) + t\, f''(t) \equiv \mathrm{d}[t\, f'(t)]$ and thus,upon integration and dividing by t, $f'(t) \geq f(t)/t - a/t$. Since $f(t)$ is non-decreasing, the right-hand side of this inequality is $\geq b$.

The function $k(t) := f'(t) - b$ is measurable, almost everywhere non-negative, and non-increasing. Hence k is locally integrable near 0 (otherwise f would be infinite at 0). In view of monotonicity $\lim_{t\to\infty} k(t)$ exists and, in view of the definition of b, it vanishes. □

□

If f is a CrF function satisfying eq. (8) then $\mathrm{D}f$ exists almost everywhere and $\mathrm{D}f = k$, $f(0) = a$ and $\lim_{t\to\infty} \mathrm{D}f(t) = b$.

Examples 3.5.

1. *The function* $f(t) := a + b\, t + c\, t^\alpha$, $a, b, c \geq 0$, $0 < \alpha < 1$, *is a CrF function, with* $k(t) = \alpha\, t^{\alpha-1}$.

2. *The function* f *defined above is not a CrF if* $\alpha > 1$.

Theorem 3.6.

(i) If $f \in \mathfrak{Q}$ *and* $g(x) := x^2\, \tilde{f}(x)$ *then* $g \in \mathfrak{B}$.

(ii) If $f \in \mathfrak{B}$ *then there is a CrF* g *such that* $f(x) = x^2\, \tilde{g}(x)$.

Proof. Ad (i)
Eq. (8) implies that f has the Laplace transform

$$\tilde{f}(p) = \frac{a}{p} + \frac{b}{p^2} + \frac{1}{p}\tilde{k}(p)$$

hence

$$p^2\, \tilde{f}(p) = a\, p + b + p\, \tilde{k}(p)$$

But

$$p\, \tilde{k}(p) = p \int_0^\infty \mathrm{e}^{-pr}\, k(r)\, \mathrm{d}r = \int_0^\infty \mathrm{d}\left[1 - \mathrm{e}^{-pr}\right]\, k(r) = -\int_0^\infty \left[1 - \mathrm{e}^{-rp}\right]\, \mathrm{d}k(r)$$

Define the Radon measure λ on $\mathbb{R}_+$ by the formula $\int \varphi(r)\,\lambda(\mathrm{d}r) = -\int \varphi(r)\,\mathrm{d}k(r)$ for continuous functions φ with compact support in $\mathbb{R}_+$. The measure λ is positive and

$$\int_{]0,\infty[} \frac{r}{1+r}\lambda(\mathrm{d}r) = -\int_{]0,\infty[} \frac{r\,\mathrm{d}k(r)}{1+r} = \int_0^\infty \frac{k(r)}{(1+r)^2}\mathrm{d}r < \infty$$

We have used the facts that $\lim_{r\to 0+} r\,k(r) = 0$ (because k is locally integrable) and k is non-increasing. Consequently the Radon measure λ satisfies inequality (3). We now have the identity

$$p\,\tilde{k}(p) = \int_0^\infty \left[1 - \mathrm{e}^{-rp}\right]\,\mathrm{d}k(r)$$

We thus see that the function g has the integral representation (4) and is thus a Bernstein function, q.e.d.

If the limit $k_0 := \lim_{r\to 0+} k(r)$ exists then ν can be defined as the Borel measure $\nu(]r,s]) = k(r) - k(s)$ for $0 \le r \le s$, provided k is redefined as a right-continuous function.

Ad (ii)

The function f satisfies eq. (4) with the Radon measure λ satisfying inequality (3). Since $r/(1+r) \ge 1/2$ for $r > 1$, it follows from eq. (3) that for every $r > 0$ the right-continuous function $k(r) := \lambda(]r,\infty[) = \int_{]r,\infty[} \lambda(\mathrm{d}\xi)$ is non-negative and finite. Since

$$\int_a^b k(r)\,\mathrm{d}r \equiv \int_a^b r\,\lambda(\mathrm{d}r) + b\,k(b) - a\,k(a)$$

is in view of eq. $(5)_1$ finite for $0 \le a < b$, the function k is locally integrable on $\overline{\mathbb{R}_+}$. It is also non-negative, non-increasing and, on account of (3), it tends to zero at infinity. The integral in eq. (4) can be integrated by parts:

$$-\int_{]0,\infty[} \left(1 - \mathrm{e}^{-pr}\right)\,\mathrm{d}k(r) = p\int_{]0,\infty[} \mathrm{e}^{-pr}\,k(r)\,\mathrm{d}r = p\,\tilde{k}(p)$$

Hence $f(t) = p^2\,\tilde{g}(p)$ with $g(t) := a\,t + b + \int_0^t k(s)\,\mathrm{d}s$ and $g \in \mathfrak{Q}$ by Theorem 3.4. $\square$

Remark 3.7. *The definition of a complete Bernstein function (Appendix A) is equivalent to the statement that, assuming again the equation* $g(x) = x^2\,\tilde{f}(x)$, $g \in \mathfrak{F}$ *if and only if* $f \in \mathfrak{B}$.

4 The wave number function in materials with a CrF creep compliance.

We now consider a viscoelastic material whose creep compliance J is a CrF.

Consider the Fourier and Laplace transform of the viscoelastic equation of motion assume

$$\rho\, p^2\, \hat{\tilde{u}} = -p\, \tilde{G}(p)\, k^2\, \hat{\tilde{u}} \tag{9}$$

and note that $p\,\tilde{G}(p) = 1/[p\,\tilde{J}(p)]$ [11]. Define the wave number function $\kappa(p)$, $p \in \mathbb{C}$, in such a way that $k = -\mathrm{i}\kappa(p)$ and $\mathrm{Re}\,\kappa(p) \geq 0$ is a solution of eq. (9).

In view of eq. (9) the wave number function can be expressed in terms of the Laplace transform $\tilde{J}$ of the creep compliance

$$\kappa(p) = \rho^{1/2}\, p^{1/2}\, \left[p^2\, \tilde{J}(p)\right]^{1/2} \tag{10}$$

If $J \in \mathfrak{Q}$ then, by Theorem 3.6, $f(p) := p^2\, \tilde{J}(p)$ is a Bernstein function. Hence $\kappa = [pf(p)]^{1/2} \in \mathcal{K}_{\mathfrak{Q}} := p^{1/2}\, \mathfrak{B}^{1/2}$, where $\mathfrak{B}^\alpha := \{f^\alpha \mid f \in \mathfrak{B}\}$. We note that $\mathfrak{B}^\alpha \subset \mathfrak{B}$ for $0 \leq \alpha \leq 1$ because f^α is a composition of two Bernstein functions $x \to x^\alpha$ and f, see Theorem 2.7. However $p\, f(p)$, $f \in \mathfrak{B}$, is in general not a Bernstein function, for example if $f(p) = 1 - \mathrm{e}^{-p}$.

In [4] it was proved that $\mathcal{K}_{\mathfrak{B}} := \mathfrak{F} \cap p^{1/2}\, \mathfrak{F}$ is the set of all wave number functions compatible with the assumption that the relaxation modulus is LICM, or, equivalently, that the creep compliance is a Bernstein function [11]. The set of Bernstein functions $\mathfrak{B}$ is a subset of $\mathfrak{Q}$. We thus expect that the set $\mathcal{K}_{\mathfrak{B}} \subset \mathcal{K}_{\mathfrak{Q}}$. In order to check this fact directly we shall need the following theorem (Proposition 7.11 in [13]):

Theorem 4.1. *If* $0 < \alpha \leq 1$ *then* $\mathfrak{F}^\alpha = \{g \in \mathfrak{F} \mid p^{1-\alpha}\, g(p) \in \mathfrak{F}\}$.

Corollary 4.2. $\mathfrak{F} \cap p^{1/2}\, \mathfrak{F} = p^{1/2}\, \mathfrak{F}^{1/2}$.

Proof. Theorem A.4 and Theorem A.3 imply that $p^{1/2}\, \mathfrak{F}^{1/2} \subset \mathfrak{F} \cap p^{1/2}\, \mathfrak{F}$.

If $f \in \mathfrak{F} \cap p^{1/2}\, \mathfrak{F}$ then $f(p) = p^{1/2}\, g(p)$ with $g \in \mathfrak{F}$. Theorem 4.1 implies that $g \in \mathfrak{F}^{1/2}$, hence $f \in p^{1/2}\, \mathfrak{F}^{1/2}$, q.e.d. □

□

It follows from Corollary 4.2 that $\mathfrak{F} \cap p^{1/2}\, \mathfrak{F} = p^{1/2}\, \mathfrak{F}^{1/2} \subset p^{1/2}\, \mathfrak{B}^{1/2}$, as expected.

Returning to the case of a creep compliance in the CrF class, note that the wavenumber function has the form $\kappa(p) = p^{1/2}\, f(p)^{1/2}$, where f has the integral representation

$$f(p) = a + b\, p + \int_{]0,\infty[} \left[1 - \mathrm{e}^{-pr}\right]\, \nu(\mathrm{d}r) \tag{11}$$

with $a, b \geq 0$ and a non-negative Radon measure ν satisfying the inequalities (5).

Using Theorem 2.6 it is now easy to prove that $\kappa(p) = b^{1/2}\, p + R(p)$, where $R(p) = \mathrm{o}[p]$ for $p \to \infty$. The phase function in Green's functions has the form

$$\exp(p\, t - \kappa(p)|x|) = \exp(p(t - b^{1/2}\, |x|) - R(p)|x|)$$

with $p = -\mathrm{i}\omega$, $\omega \in \mathbb{R}$. The arguments used in the proof of Theorem 7.2 in [4] can be used to demonstrate that the Green's function vanishes for $|x| > b^{-1/2}\, t$, hence $b^{-1/2}$ is the wavefront speed.

5 Main theorem.

Behavior of the attenuation and dispersion function for real frequencies (imaginary p) is of particular interest. We shall therefore examine the behavior of the analytic continuation of a Bernstein function $f(p)$ on the imaginary axis.

Analytic continuation of f to the imaginary axis yields a function $F(\omega) := f(-\mathrm{i}\omega)$, $\omega \in \mathbb{R}$. $f_{\mathrm{R}}(\omega) := \mathrm{Re}\, F(-\mathrm{i}\omega)$ and $f_{\mathrm{I}}(\omega) := \mathrm{Im}\, F(-\mathrm{i}\omega)$. Hence

$$f_{\mathrm{R}}(\omega) = a + \int_{]0,1]} [1 - \cos(r\omega)]\, \nu(\mathrm{d}r) + \int_{]1,\infty[} [1 - \cos(r\omega)]\, \nu(\mathrm{d}r) \tag{12}$$

$$f_{\mathrm{I}}(\omega) = \int_{]0,1]} \sin(r\omega)\, \nu(\mathrm{d}r) + \int_{]1,\infty[} \sin(r\omega)\, \nu(\mathrm{d}r) \tag{13}$$

Lemma 5.1. *The integrals in eqs. (12-13) are bounded by linear functions of the variable ω.*

Proof. Note that $|\sin(x)/x| \leq 1$ (the numerator and the denominator vanish at 0 and the absolute value of the derivative of the numerator does not exceed the value of the derivative of the numerator equal to 1). Hence $|\sin(r\omega)/r| \leq |\omega|$. In the same way one shows that $|(1 - \cos(r\omega)/r| \leq |\omega|$. Hence

$$\left| \int_{]0,1]} \sin(r\omega)\, \nu(\mathrm{d}r) \right| = \left| \int_{]0,1]} \frac{\sin(r\omega)}{r}\, r\nu(\mathrm{d}r) \right| \leq |\omega| \int_{]0,1]} r\, \nu(\mathrm{d}r)$$

and

$$\left| \int_{]0,1]} [1 - \cos(r\omega)]\, \nu(\mathrm{d}r) \right| = \left| \int_{]0,1]} \frac{1 - \cos(r\omega)}{r}\, r\nu(\mathrm{d}r) \right| \leq |\omega| \int_{]0,1]} r\nu(\mathrm{d}r)$$

Furthermore

$$\left| \int_{]1,\infty[} \sin(r\omega)\, \nu(\mathrm{d}r) \right| \leq \int_{]1,\infty[} \nu(\mathrm{d}r) < \infty$$

$$\left| \int_{]1,\infty[} [1 - \cos(r\omega)]\, \nu(\mathrm{d}r) \right| \leq 2 \int_{]1,\infty[} \nu(\mathrm{d}r) < \infty$$

Hence $|f_{\mathrm{R}}(\omega)| \leq a + C|\omega|$ and $|f_{\mathrm{I}}(\omega)| \leq b + | \leq D|\omega|$ for some positive reals a, b, C, D. $\square$

Theorem 5.2.

$$|\kappa_{\mathrm{I}}(\omega)| \leq K + L\, |\omega| \tag{14}$$

$$|\kappa_{\mathrm{R}}(\omega)| \leq K + L\, |\omega| \tag{15}$$

for some real constants K and L.

Proof. If $m = \max\{a, C\}$, $M = \max\{b, D\}$ then

$$|F(\omega)| = \sqrt{f_{\mathrm{R}}(\omega)^2 + f_{\mathrm{I}}(\omega)^2} \leq \sqrt{2}\,(m + M\,|\omega|)$$

and therefore

$$|\kappa(-\mathrm{i}\omega)| \leq 2^{1/4}\,\left(m\,|\omega| + M\,|\omega|^2\right)^{1/2} \leq 2^{1/4}\,\left(N + M^{1/2}\,|\omega|\right)$$

where $N := m/(2M^{1/2})$. The thesis follows with $K = 2^{1/4}\,N$ and $L = 2^{1/4}\,M^{1/2}$. □

It follows that the attenuation is majorized by a linear function of frequency. In the high frequency range it increases at most at a linear rate. This bound can perhaps be sharpened.

6 Concluding remarks.

It has been shown that the assumption that the creep compliance is non-decreasing and concave implies that the attenuation function increases at a sublinear rate in the high-frequency range. This result is somewhat weaker than for materials with completely monotonic relaxation moduli and Bernstein class creep compliances [8, 4].

This result shows that some acoustic wave equations applied to ultrasonic tests of materials and especially for ultrasound in bio-tissues still popular in literature [5, 6, 7] are not compatible with viscoelasticity. Such equations are based on the assumption of a power law attenuation with an exponent exceeding 1. We have thus shown that such equations are incompatible with the assumption that the material is linear viscoelastic and the creep compliance is non-decreasing and concave.

The results obtained in this paper are compatible with experimental data. In the frequency range covered by measurements the frequency dependence of attenuation is represented by a power law with an exponent $1 \leq \alpha \leq 2$ over less than three decades of frequency expressed in MHz. This fact is consistent with the results obtained in [4] for low-frequency attenuation in viscoelastic solids. It should however be borne in mind that the sound absorption measured in heterogeneous media includes loss due to scattering which is here not accounted for.

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A

Definition A.1. *A real function f on $[0, \infty[$ is said to be a complete Bernstein function if there is a Bernstein function g such that $f(x) = x^2\, \tilde{g}(x)$.*

$$f(x) = a + b\, x + x \int_{]0,\infty[} \frac{\mu(\mathrm{d}r)}{x + r} \tag{16}$$

Theorem A.2. *[17]*
Every CBF can be continued to an analytic function F on $\mathbb{C}_-$ satisfying the following conditions:

(i) *F assumes real values on $\mathbb{R}_+$;*

(ii) *a finite limit $\lim_{\substack{x\to 0\\ x>0}} F(x)$ exists and is non-negative;*

(iii) $\operatorname{sgn}(\operatorname{Im}[F(z)]) = \operatorname{sgn}(\operatorname{Im}[z])$*;*

(iv) *The function F has an unique integral representation*

$$F(z) = a + b\, z + \int_{]0,\infty[} \frac{tz - 1}{t + z} \rho(\mathrm{d}t)$$

with $a, b \geq 0$ and a positive Radon measure ρ satisfying the inequality satisfying the inequality

$$\int_{]0,\infty[} \frac{\rho(\mathrm{d}t)}{1 + t} < \infty$$

If a complex valued function F satisfies either the conditions (i)–(iii) or condition (iv) then the restriction of F to $\overline{\mathbb{R}_+}$ is a CBF.

This theorem has the following corollary:

Theorem A.3. *If f is a CBF and $0 \leq \alpha \leq 1$ then $f(x)^\alpha$ is also a CBF.*

Theorem A.4. *If $f, g \in \mathfrak{F}$ and $0 \leq \alpha \leq 1$ then the pointwise product $h := f^\alpha\, g^{1-\alpha} \in \mathfrak{F}$.*

Proof. The functions f, g map $\mathbb{R}_+$ into $\overline{\mathbb{R}_+}$, hence h has the same property.

The functions f, g are holomorphic in $\mathbb{C}_\pm$ and map $\mathbb{C}_\pm$ into $\overline{\mathbb{C}_\pm}$, hence $h(z)$ is holomorphic in $\mathbb{C}+$ and $0 \leq \arg h(z) \leq \alpha\, \pi + (1 - \alpha)\, \pi = \pi$. By Theorem A.2 $h \in \mathfrak{F}$. $\square$